Math 208 Midterm II

May 21, 2025

NAME:

UW EMAIL:

STUDENT ID NUMBER:

1	/12
2	/12
3	/12
4	/12
5	/12
Bonus	/2
Total	/60

Instructions. [1]

- For each problem below, give a carefully explained solution using the vocabulary and notation from class. A correct answer with no supporting work or explanation will receive a zero.
- Put a $\boxed{\text{box}}$ around your final answers.
- If a solution involves a numerical answer, collect all terms and reduce all fractions. No decimal expansion is necessary.
- You are allowed a simple calculator and notesheet. Other notes, electronic devices, etc, are not allowed. Take a few pencils from your pencil case and put all other items away for the duration of the exam.
- All the questions can be solved using (at most) simple arithmetic. (If you find yourself doing complicated calculations, there might be an easier solution.)
- Raise your hand if you have any questions or spot a possible error.
- Please stay in your seat until all exams are collected and the class is dismissed.

Good luck!

[1] Test code: 8745

(1) Define matrices

$$A = \begin{bmatrix} 2 & 2 \\ 0 & 0 \\ 8 & 2 \\ 0 & 5 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 8 \\ 0 & 1 \end{bmatrix}.$$

(a) (2 points) What is the rank and nullity of A? Justify your answer.

(b) (2 points) What is the rank and nullity of A^T? Justify your answer.

(c) (4 points) Compute $AB^2 - AB$.

(d) (4 points) Is $AB^2 - AB$ invertible? If so, find its inverse and verify your claim by matrix multiplication. If not, find a nonzero vector $\mathbf{v}$ such that $(AB^2 - AB)\mathbf{v} = \mathbf{0}$.

(2) Let C be the matrix

$$C = \begin{bmatrix} 3 & -7 & 1 & 1 & 0 & 6 \\ 0 & 0 & -2 & 4 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}.$$

(a) (3 points) Find a basis for the column space of C. Justify your answer.

(b) (3 points) Find a basis for the row space of C. Justify your answer.

(c) (6 points) Find a basis for the null space of C. Justify your answer.

(3) Consider the following row equivalent matrices:

$$A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 \end{bmatrix} \sim \begin{bmatrix} 3 & 0 & -1 & 5 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 & \mathbf{b}_4 \end{bmatrix}$$

Let $\mathrm{col}(A)$ be the column space of A. Answer the following with reasons.

(a) (4 points) State a rule you can use to determine if any set of linear combinations of columns of the matrix A is a basis for $\mathrm{col}(A)$.

(b) (8 points) Apply the rule you stated above to identify which of the following sets form a basis for $\mathrm{col}(A)$ by saying "YES because ..." or "NO because ..." in each case.

Is $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ a basis for $\mathrm{col}(A)$?

Is $\{\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_4\}$ a basis for $\mathrm{col}(A)$?

Is $\{\mathbf{a}_1 + 2\mathbf{a}_4, \mathbf{a}_3 + \mathbf{a}_4\}$ a basis for $\mathrm{col}(A)$?

Is $\{\mathbf{a}_1, \mathbf{a}_3 + 2\mathbf{a}_4\}$ a basis for $\mathrm{col}(A)$?

(4) Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be the linear transformation that maps the standard basis vectors in $\mathbb{R}^3$ to the following vectors in $\mathbb{R}^2$:

$$e_1 \to \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad e_2 \to \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad e_3 \to \begin{bmatrix} 3 \\ 0 \end{bmatrix}.$$

(a) (4 points) What is the matrix A such that $T(v) = Av$? Verify your transformation by showing the computations of $Ae_1, Ae_2,$ and Ae_3 and confirming they agree with T.

(b) (4 points) Find a basis for the kernel and range of T? Justify your answers.

(c) (4 points) Let $C = \{(x, y, z) : 0 \leq x, y, z \leq 1\}$ be the $1 \times 1 \times 1$ cube with one corner at the origin in $\mathbb{R}^3$. Sketch the image of the cube C under the linear transformation T labeling any important features of the image.

(5) When Sally the strawberry farmer bought her farm, there were several survey marks on the ground in a square grid, which she used to note the positions (x, y) of all of the irrigation valves she installed for watering the stawberry plants. The survey mark near her house is what she used as $(0, 0)$. Once the strawberry plants started to grow, it was hard to find the survey marks, so the following year she wrote down the coordinates of the irrigation valves using the rows and columns of her stawberry fields, which travel away from her house on a grid parallel to the two roads bordering her house using vectors

$$\begin{bmatrix} -0.2 \\ 0.1 \end{bmatrix}, \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}.$$

This year, she wants to plant the fields in a new pattern to better align with the view of the mountains. The new grid will be based on the vectors

$$\begin{bmatrix} -0.5 \\ 0.7 \end{bmatrix}, \begin{bmatrix} 0.3 \\ 0.4 \end{bmatrix}.$$

(a) (6 points) What function can Sally use to compute the locations of all of her irrigation valves in terms of the new grid from the coordinates she used last year?

(b) (6 points) In the old grid coordinates, the main shut-off valve was at $(20, 40)$, where is it in the new grid coordinates?

Bonus: Let U and V be two subspaces of $\mathbb{R}^n$.

(a) (1 points) Under what conditions is the intersection of U and V a subspace of $\mathbb{R}^n$? Justify your answer. (Intersection means the set of all vectors in both U and V.)

(b) (1 points) Under what conditions is the union of U and V a subspace of $\mathbb{R}^n$? Justify your answer. (Union means the set of all vectors in either U or in V or in both.)